

Math Challenge Club September 12: Counting

Today we will learn how to count in mathematical objects and other things such as handshakes or fence posts or balls...

Counting Whole Numbers

Lets start with something that is rather easy, but also easy to do wrong:

1. Eva takes a pack of index cards and numbers them starting with 10 and ending with 50. How many cards are in her pack?
2. How many whole numbers are less than 600 but greater than 500?
3. Jonathan starts counting at 130 and counts by fives. What is the 13th number that Jonathan says?
4. Claire orders new checkbooks. Checks are numbered sequentially. If she orders 400 checks and the last check is numbered 3474, what is the number on the first check?

Here are a couple of rules that can be useful for this type of problems:

Let a and b be whole numbers with a less than b . How many whole numbers are there between a and b *exclusive*?

How many whole numbers are there between a and b *inclusive*?

Counting Fence Posts

Jeremy owns a rectangular plot of land that is 60 meter long and 30 meter wide.

1. If he places fence posts at each corner and posts are placed every three meter, how many fence posts will there be on each side?
2. If he places fence posts at each corner and posts are placed every three meter, how many fence posts will he need to enclose his land?

Here is a list of related problems for you to solve:

1. You are assigned the even problems from 10 to 40 for tonight's homework. How many problems is this?
2. Chopping a carrot into slices (the usual way), how many cuts are required to make 20 pieces?
3. How many even perfect squares are there from 100 to 10,000 inclusive?
4. The circumference of a circular table is 30 feet. If a set of silverware is placed every 3 feet around the circumference of the table, how many place settings are there?

5. How many distinct pairs of consecutive integers have a product less than 10,000?
6. How many whole numbers less than 100 are multiples of 3, but not multiples of 5?
7. A field in the shape of a regular hexagon is enclosed with a fence using 120 evenly spaced spots. Assuming that there is a spot at every vertex (corner), how many posts are on each side of the hexagon?
8. We wish to place natural numbers around a circle such that the absolute values of the differences of the pairs of neighbouring numbers are all different.
 - (a) Is it possible to place the numbers from 1 to 2009 in this manner? [Hint: First try this with a smaller number.]
 - (b) Is it possible to remove one of the numbers numbers from 1 to 2009 in such a way that the remaining numbers can be placed in this way?

Venn Diagrams

Venn diagrams are helpful if you have to count items or people who belong to several groups, but don't belong to other groups. When you fill in a Venn diagram, always start from the inside out; with the smallest set that belongs to the most groups and then work your way out. Here is an example to practice this with:

In a science classroom: 19 students have a brother, 15 students have a sister, 7 students have both a brother and a sister, and 6 students don't have any siblings at all. How many students are in the classroom?

This problem had only two groups (with their intersection and the remainder). It gets a bit harder when you do this with more.

Every student who applied for admission to a veterinary school has at least one pet: 30 have a cat, 28 have a dog, and 26 have a fish. If 13 students have a fish and a cat, 15 students have a fish and a dog, 11 students have a cat and a dog and 4 students have a cat, a dog and a fish. How many students applied to veterinary school?

Do this first with a Venn diagram.

Then lets talk about how we could do this with a formula and a calculation. We will rediscover what people have called the *inclusion-exclusion principle*.

Here are some practice problems:

1. At the pound there are 40 dogs. If 22 dogs have spots and 30 dogs have short hair, what is the fewest number of dogs that can have short hair and spots?
2. Ten friends go out to dinner together: 7 order an appetizer, 5 order soup, and 4 order a salad. If everyone orders something, but no one orders exactly two things, how many people order all three things?
3. How many of the first 729 positive integers are perfect squares, cubes, or both?
4. An auto dealership sells expensive foreign automobiles. You are looking for a black convertible Porsche, but the website lets you only look for one or two criteria at once. Also, the sales person tells you that the lot has 50 cars that meet at least one of your criteria. On the website you find that: there are 18 Porsches, 25 black cars and 16 convertibles. Also, there are 3 black convertibles, 4 black Porsches, and 5 convertible Porsches. How many black convertible Porsches are there?

Bowling Pins and Handshakes

How many bowling pins are set up in a standard bowling lane? Draw the pattern of the pins. Now suppose that you add a fifth row. How many pins do you have now? Suppose that there were 100 rows of pins. Find a systematic way to count them so that you can do this for any number of rows.

Here is a problem that may look very different, but we will see that it is related:

Suppose that 12 strangers get together for a hike and each person shakes each other person's hand exactly once. How many handshakes have occurred? Can you make a more general formula for n people?

Here are some related practice problems:

1. We have five points lying on a half circle. How many straight line segments do you need to connect each point to the other four points?
2. Suppose that you have 30 points in the plane and no line passes through three or more of these points. How many segments will you need to connect these 30 points (so that each point is connected to each other point by a segment)?
3. There are two diagonals in a square. A pentagon has five diagonals and a hexagon has nine diagonals. How many diagonals are there in a regular polygon with 15 sides?

4. Several couples arrive at a dinner party and each person shakes the hand of every other person at the party, except for his or her spouse. If there were a total of 112 handshakes, how many couples attended the party?
5. What is the largest prime factor of $1 + 2 + 3 + \cdots + 60$?

Double Counting

A very useful trick if you need to prove something about the things that you have counted, is to count them twice in different ways. Here is a good example of this:

The Handshaking Lemma At a party some people shake hands. Each pair of people shake hands at most once. In that case the number of people who shake hands an odd number of times is even.

Count the hand shakes in two different ways to give an argument for this result.

Here are a couple of harder problems:

1. The Dwarves in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the Pippin, with gold coins to the value of 1 Pippin, 10 Pippins, 100 Pippins, and 1000 Pippins. In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 Pippins?
2. The seven dwarfs decide to form four teams to compete in the Millenium Quiz. Of ourse, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy, and one of Bashful and Sneezy. In how many ways can the four teams be made up? (the order of the teams and the order of the dwarfs on a team does not matter, but each dwarf must be on exactly one team.)
Suppose that Snow White agrees to take part as well. In how many ways can you now form teams?
3. Twelve people are seated arond a circular table. In how many ways can six pair of people engage in handshakes so that no arms cross? (nobody is allowed to shake hands with more than one person at once)
4. Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz. However, he wants to avoid putting any pair of twins into the same team. Subject to this condition:
 - (a) In how many ways can he split them into two teams of six?
 - (b) In how many ways can he split them into three teams of four?