Math Challenge Club — Notes Jan. 12 —Counting techniques

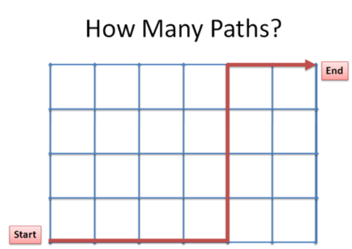
Addition and Multiplication Principle

A doughnut shop has five types of doughnuts, four types of muffins. You receive a coupon which gives you a free doughnut or muffin. How many possibilities? What if the coupon gives you a free doughnut **and** a muffin. How many choices?

This question illustrates the *multiplication* and *addition* principle. If you have to choose an element from one set (doughnuts) *or* another set (muffins), then the choices add. So there are 5+4=9 ways to choose a dougnut or muffin. If your choose one element from one set *and* an element from another set, then the choices multiply (each choice from the first set can be combined with each choice from the second set). So there are $5 * 4 = 20$ ways to choose a doughnut and a muffin.

Grid paths

How many ways are there to walk along a grid from the point $(0, 0)$ to the point $(5, 3)$? We will include only path that consist of steps to the right (R) –one step in the x-direction—and up (U)—one step in the y-direction. Thus, we only count shortest paths along the grid. Below is an example of a grid path from $(0, 0)$ to $(6, 4)$.



Let us start build up the solution starting with paths that are easier to count. Use the notation $p(a, b)$ for the number of paths from $(0, 0)$ to (a, b). Some easy values: $p(1, 0) = p(0, 1) = 1$, $p(1, 1) = 2$. In fact, if the endpoint is on the x-axis or the y-axis then there is no choice in how to get there: only right steps for an endpoint on the

x-axis, and only up steps for an endpoint on the y-axis. This translates into the rule: For any positive integer (whole number) a:

$$p(a, 0) = p(0, a) = 1.$$

Now consider $p(2, 1)$. Any path to the point $(2, 1)$ either goes through the point $(1, 1)$ or the point $(2, 0)$ (*Draw the picture!*). There are $p(1, 1) = 2$ paths from $(0, 0)$ to $(1, 1)$, and $p(2, 0) = 1$ paths from $(0, 0)$ to $(2, 0)$. Adding these gives the number of paths from $(0, 0)$ to $(2, 1)$, since *any* path from $(0, 0)$ to $(2, 1)$ must start out as a path from $(0, 0)$ to $(1, 1)$ *or* a path from $(0, 0)$ to $(2, 0)$. So $p(2, 1) = p(1, 1) + p(2, 0) = 2 + 1 = 3$. Continuing this argument, we can compute $p(3, 1)$ by adding $p(2, 1)$ and $p(3, 0)$: $p(3, 1) = 3 + 1 = 4$. And so, $p(4, 1) = p(3, 1) + p(4, 0) = 4 + 1 = 5$, and $p(5, 1) = 5 + 1 = 6$, and so on.
Our next observation is that the number of paths to a point (a, b) is the same as the number of paths to the point (b, a): just turning the picture will turn up moves into right moves and vice versa. So $p(1, 2) = p(2, 1) = 3$. Now we can compute $p(2, 2) = p(2, 1) + p(1, 2) = 3 + 3 = 6$. In fact, we can express a general formula, which will allow us to compute all values $p(a, b)$:

$$p(a, b) = p(a - 1, b) + p(a, b - 1).$$

1. How many ways are there to walk along a grid from the point $(0, 0)$ to the point $(5, 3)$?

2. How many ways are there to walk along a grid from the point $(1, 1)$ to the point $(5, 3)$?

3. How many ways are there to walk along a grid from the point $(3, 1)$ to the point $(5, 3)$?

4. How many ways are there to walk along a grid from the point $(0, 0)$ to the point $(5, 3)$ so that you visit the point $(2, 2)$? *Hint: each path from* $(0, 0)$ *to* $(5, 3)$ *via* $(2, 2)$ *consists of a path from* $(0, 0)$ *to* $(5, 3)$ *AND a path from* $(2, 2)$ *to* $(5, 3)$.

5. How many ways are there to walk along a grid from the point $(0, 0)$ to the point $(5, 3)$ so that you visit the point $(2, 1)$ OR the point $(1, 2)$?

6. How many ways are there to walk along a grid from the point $(0, 0)$ to the point $(5, 3)$ so that you DO NOT visit the point $(3, 1)$?

Pascal's triangle

When you rearrange all the numbers $p(a, b)$ so that all numbers with the same sum $a + b$ are in the same row, then we get *Pascal's triangle.*

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

The numbers in Pascal's triangle are called the *binomial coefficients*. The notation is as follows: the $k-th$ entry in the $n-th$ row is denoted as $\binom{n}{k}$. They correspond to the path numbers in the following way:

$$p(a,b) = \binom{a+b}{a}.$$

We can also think of it as: $\binom{n}{k}$ is the number of grid paths from $(0,0)$ with k right moves and n steps in total (and thus $n-k$ up moves.

7. Add all the binomial coefficients in each row. What pattern do you observe? Can you think of an explanation?

Representing paths as words

Instead of drawing the picture, we can also represent a path by listing all the moves in order, using R for a right move, and U for an up move. For example, the path from our example earlier (see picture on page 1) can be represented as: RRRRUUUURR.

Every path from $(0,0)$ to $(5,3)$ corresponds to a "word" with 5 R's and 3 U's, and vice versa. Therefore, $p(5,3)$ also represents the number of words with 5 R's and 3 U's.

In terms of binomial coefficients, we see that $\binom{n}{k}$ equals the number of words of length n with exactly k R's.

This can help answer a different type of question:

8. Aaron, Charly and Debby have ten candies between them (the candies are identical). In how many ways can the candies be divided between the three?

   The answer follows if we realize that we can represent each division by a word of ten zeros and two ones. The number of zeros before the first one represents the candies that Aaron gets, the number of zeros between the two zeros is the number of candies that Charly gets, and the number of zeros after the last one represents the candies that Debby gets. For example, the solution represented by 000100000010 is where Aaron gets 3 candies, Charly gets 6 and Debby gets 1.

9. How can we represent the solution where Aaron and Debby get 4 candies, and Charly gets one?

10. What solution is represented by 010000000001? And by 110000000000 ?

    Clearly, each word with two ones and ten zeros represents a solution, and vice versa. Thus, the number of solutions equals the number of words with two ones and ten zeros. Change zeros into R's and ones into U's, and we see that this equals the number of paths from $(0, 0)$ to $(10, 2)$. Thus, we can bring this problem back to the numbers we know.

11. How many ways can Aaron, Charly and Debby divide the candies so that each of them gets at least two candies?

12. How many ways can Aaron, Charly and Debby divide the candies so that Aaron does not get more than seven candies?